Proximity to a Traditional Physical Store: The Effects of Mitigating Online Disutility Costs

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October 22, 2018

ONLINE APPENDIX

Proofs

Without loss of generality, we assume $t = 1$ in our proofs. Note that the value of t only scales the problem. Also, for exposition, we define the following:

Definition 1 *1 $L = ac$, $U = 1/2 - a/2 + ac$, and $U' = [1 - a + 2ac + 4c]/6$.*

Using Definition 1 and given conditions on μ defined in Section 2.1, we have

- In b model, $\mu \in (L, 1/2)$.
- In se model, $\mu \in (L, U)$.
- In br model, $\mu \in (L, U')$.

Proof of Proposition 1 When both b and br models are considered together, from Definition 1, we obtain $\mu \in (L, U')$. In comparing π_r^{br} and π_r^b, we have

$$\pi_r^b - \pi_r^{br} = \kappa_1\mu^2 + \kappa_2\mu + \kappa_3,$$

where

$$\kappa_1 = \frac{16a + 20}{72[a-1]}, \quad \kappa_2 = \frac{u[-72ac + 8a - 8]}{72[a-1]}, \quad \text{and } \kappa_3 = \frac{20a^2c^2 - 8a^2c - a^2 + 16ac^2 + 8ac + 2a - 1}{72[a-1]}.$$

Thus, with respect to μ, $\pi_r^b - \pi_r^{br}$ is a concave parabola.

Since $\pi_r^{br} - \pi_r^b$ is continuous in μ, we can derive that when $a \in (0,1)$ and $c \in (1/8, 1/2)$,

$$\begin{aligned}
[\pi_r^b - \pi_r^{br}]\Big|_{\mu \to U'} &= [\pi_r^b - \pi_r^{br}]\Big|_{\mu = U'} \\
&= \frac{1}{162}[1-a][1-2c][2ac - a + 10c + 4] \\
&> 0
\end{aligned}$$

$$\text{and } [\pi_r^b - \pi_r^{br}]\Big|_{\mu \to L} = [\pi_r^b - \pi_r^{br}]\Big|_{\mu = L} = \frac{[1-a][1-16ac^2]}{72}.$$

Hence, we can conclude:

- If $a \leq 1/[16c^2]$, $[\pi_r^b - \pi_r^{br}]\Big|_{\mu \to L} \geq 0$, which implies that $\pi_r^b > \pi_r^{br}$ for all μ in the feasible region, $\mu \in (L, U')$.
- If $a > 1/[16c^2]$, $[\pi_r^b - \pi_r^{br}]\Big|_{\mu \to L} < 0$, which implies that there is a $\mu^* \in (L, U')$ such that
  - if $\mu < \mu^*$, then $\pi_r^b < \pi_r^{br}$;
  - if $\mu > \mu^*$, then $\pi_r^b > \pi_r^{br}$;
  - otherwise if $\mu = \mu^*$, then $\pi_r^b = \pi_r^{br}$.

  μ^* is the smaller solution of $\pi_r^b = \pi_r^{br}$; and,

  $$\mu^* = \frac{-\sqrt{-80a^3c^2 + 32a^3c + 4a^3 + 160a^2c^2 - 64a^2c + a^2 - 80ac^2 + 32ac - 14a + 9} + 18ac - 2a + 2}{2[4a+5]}.$$

Q.E.D.

Proof of Corollary 1 In comparing π_r^b and π_r^{se}, we have

$$\pi_r^{se} - \pi_r^b = \kappa_1 \mu^2 + \kappa_2 \mu + \kappa_3,$$

where

$$\kappa_1 = -\frac{[16a + 20]}{72[a-1]}, \quad \kappa_2 = -\frac{u[-72ac + 8a - 8]}{72[a-1]}, \quad \text{and } \kappa_3 = -\frac{36a^2c^2 - 18a^2 + 19a - 1}{72[a-1]}.$$

Observe that $\pi_r^{se} - \pi_r^b$ is a convex parabola with respect of μ, where $\mu \in (L, \min\{1/2, U\})$. Denoting the axis of symmetry as Ω, we get

$$\Omega = -\frac{\kappa_2}{2\kappa_1} = \frac{9ac - a + 1}{4a + 5}.$$

When $c \leq 1/2$, $U \leq 1/2$ which means $\mu \in (L, U)$. Note $\pi_r^{se} - \pi_r^b$ is continuous in μ and the value of $\pi_r^{se} - \pi_r^b$ at the upper bound of μ, $\mu \to U$, is

$$\begin{aligned} [\pi_r^{se} - \pi_r^b]\Big|_{\mu \to U} &= [\pi_r^{se} - \pi_r^b]\Big|_{\mu = U} \\ &= \frac{1}{72} a \left[-4a[1-2c]^2 - 24c + 21\right] \\ &> a \left[-4 - 24 \times \frac{1}{2} + 21\right] > 0. \end{aligned}$$

When $a \in (0,1)$ and $c \in (1/4, 1/2]$, we obtain $L < \Omega < U$. Thus, if $[\pi_r^{se} - \pi_r^b]|_{\mu=\Omega} \geq 0$, we have $\pi_r^{se} - \pi_r^b \geq 0$ for $\mu \in (L, U)$. Otherwise, there are two solutions for $\pi_r^{se} - \pi_r^b = 0$, denoted by μ_1^* and μ_2^*, where

$$\mu_1^* = \frac{-3\sqrt{-[a-1]\left[8a^2\left[2c^2 - 1\right] + 2a[4c-5] + 1\right]} + 2a[9c-1] + 2}{8a + 10}$$

$$\text{and } \mu_2^* = \frac{3\sqrt{-[a-1]\left[8a^2\left[2c^2-1\right]+2a[4c-5]+1\right]}+2a[9c-1]+2}{8a+10}.$$

Since $[\pi_r^{se} - \pi_r^b]|_{\mu=U} > 0$, we have $L < \Omega < \mu_2^* < U$. Hence, we can conclude:

- When $\mu_1^* \leq L$,
  - if $\mu < \mu_2^*$, $\pi_r^{se} - \pi_r^b < 0$;
  - if $\mu > \mu_2^*$, $\pi_r^{se} - \pi_r^b > 0$;
  - if $\mu = \mu_2^*$, $\pi_r^{se} - \pi_r^b = 0$.
- When $\mu_1^* > L$,
  - if $\mu_1^* < \mu < \mu_2^*$, $\pi_r^{se} - \pi_r^b < 0$;
  - if $\mu < \mu_1^*$ or $\mu > \mu_2^*$, $\pi_r^{se} - \pi_r^b > 0$;
  - if $\mu = \mu_1^*$ or $\mu = \mu_2^*$, $\pi_r^{se} - \pi_r^b = 0$.

When $c > 1/2$, $U > 1/2$ which means $\mu \in (L, 1/2)$. It is difficult to definitively characterize the difference in π_r^b and π_r^{se}. However, the general trend continues. Q.E.D.

Proof of Proposition 2 *(i)* From the total costs to consumers in (4) and (15),

$$\frac{\partial[\omega^s - \omega^{se}]}{\partial c} = \frac{a(a - ac - t + \mu)}{a - t} \quad \text{and} \quad \frac{\partial^2[\omega^s - \omega^{se}]}{\partial c^2} = \frac{-a^2}{a-t} > 0$$

so that $[\omega^s - \omega^{se}]$ is convex. In addition, $[\omega^s - \omega^{se}]$ has no real roots, but only complex roots. Hence, $\omega^s > \omega^{se}$.

(ii) From the total costs to consumers in (15) and (23), and using the constraints in (13) for the Salop model with online stores together with those in (20) for the Balasubramanian model with retailers selling online:

$$\omega^{se} - \omega^{br} = 17 - 20c + 8c^2 > 0 \;\; \forall \;\; c \in [1/4, 1/2].$$

Q.E.D.

Proof of Proposition 3 In comparing γ^{se} and γ^s, we have

$$\gamma^{se} - \gamma^s = \kappa_1\mu^2 + \kappa_2\mu + \kappa_3,$$

where

$$\kappa_1 = \frac{3}{2[a-1]}, \quad \kappa_2 = \frac{-24ac + 8a - 8}{8[a-1]}, \quad \text{and } \kappa_3 = \frac{12a^2c^2 - 8a^2c + a^2 + 8ac - 2a + 1}{8[a-1]}.$$

Thus, with respect to μ, $\gamma^{se} - \gamma^s$ is a concave parabola. Denoting the axis of symmetry as Ω, we get

$$\Omega = \frac{1}{3}[3ac - a + 1].$$

Using Definition 1, we can show $L < \Omega < U$. At $\mu = \Omega$, we have

$$\gamma^{se} - \gamma^{s}\Big|_{\mu=\Omega} = \frac{1-a}{24} > 0.$$

Because $\gamma^{se} - \gamma^{s}$ is continuous in μ, we obtain

$$[\gamma^{se} - \gamma^{s}]\Big|_{\mu\to L} = [\gamma^{se} - \gamma^{s}]\Big|_{\mu=L} = \frac{a-1}{8} < 0,$$

$$\text{and } [\gamma^{se} - \gamma^{s}]\Big|_{\mu\to U} = [\gamma^{se} - \gamma^{s}]\Big|_{\mu=U} = 0.$$

Hence, we can conclude that there is a μ^* in (L, U), such that

- if $\mu > \mu^*$, $\gamma^{se} > \gamma^{s}$;
- if $\mu < \mu^*$, $\gamma^{se} < \gamma^{s}$;
- if $\mu = \mu^*$, $\gamma^{se} = \gamma^{s}$.

μ^* is the smaller solution of equation $\gamma^{se} = \gamma^{s}$; and, $\mu^* = \frac{1}{6}[6ac - a + 1]$. Q.E.D.

Proof of Proposition 4 Directly from (5) and (10), if $t < 5\mu$ then $\gamma^{b} > \gamma^{s}$. Q.E.D.

Proof of Proposition 5 Part 1 (Traditional retail prices)

In comparing p_r^{se} and p_r^{b}, we have

$$p_r^{se} - p_r^{b} = \frac{1}{12}[6a - 6ac - 1 + 2\mu],$$

which is an increasing linear function of μ.

From Definition 1 we have that when b and se models are considered together, $\mu \in (L, \min\{\frac{1}{2}, U\})$. Note when $c \leq 1/2$, $U \leq 1/2$; otherwise, if $c > 1/2$, $U > 1/2$.

Case 1: $c \leq 1/2$. If $c \leq 1/2$, we have $\mu \in \{L, U\}$. Because $p_r^{se} - p_r^{b}$ is continuous in μ, we have

$$[p_r^{se} - p_r^{b}]\Big|_{\mu\to U} = [p_r^{se} - p_r^{b}]\Big|_{\mu=U} = \frac{1}{12}a[5 - 4c] > 0.$$

At the lower bound, we have

$$[p_r^{se} - p_r^{b}]\Big|_{\mu\to L} = [p_r^{se} - p_r^{b}]\Big|_{\mu=L} = \frac{1}{12}[-4ac + 6a - 1].$$

Thus, we can conclude that when $-4ac+6a-1 \geq 0$, $[p_r^{se}-p_r^b]\Big|_{\mu \to L} \geq 0$. Accordingly, since $p_r^{se}-p_r^b$ is increasing in μ, we can conclude that $p_r^{se}-p_r^b > 0$ for all $\mu \in (L,U)$. In contrast, if $-4ac+6a-1 < 0$, $[p_r^{se}-p_r^b]\Big|_{\mu \to L} < 0$.

Thus, the sign of $p_r^{se}-p_r^b$ is different when μ is at the lower bound and the upper bound, which means that there is a $\mu^* \in (L,U)$, such that

- if $\mu > \mu^*$, $p_r^{se} > p_r^b$;
- if $\mu < \mu^*$, $p_r^{se} < p_r^b$;
- if $\mu = \mu^*$, $p_r^{se} = p_r^b$.

μ^* is the solution of equation $p_r^{se} = p_r^b$; and, $\mu^* = \frac{1-6a+6ac}{2}$.

Case 2: $c > 1/2$. If $c > 1/2$, we have $\mu \in (L, 1/2)$. Since

$$[p_r^{se}-p_r^b]\Big|_{\mu \to \frac{1}{2}} = \frac{1}{2}a[1-c],$$

we have that if $c \leq 1$, $p_r^{se}-p_r^b > 0$ at $\mu = 1/2$; otherwise if $c > 1$, $p_r^{se} < p_r^b$ at $\mu = 1/2$.

When $c \in (1/2, 1]$, similar to Case 1, if $a \geq 1/[6-4c]$, we have $p_r^{se}-p_r^b > 0$ for all $\mu \in (L, 1/2)$. Otherwise if $a < 1/[6-4c]$, we can solve $p_r^{se} = p_r^b$ for μ^* such that if $\mu > \mu^*$, $p_r^{se} > p_r^b$; if $\mu < \mu^*$, $p_r^{se} < p_r^b$; if $\mu = \mu^*$, $p_r^{se} = p_r^b$.

When $c > 1$, $p_r^{se}-p_r^b < 0$ at $\mu = 1/2$. Because $p_r^{se}-p_r^b$ is increasing when $\mu \in (L, 1/2)$, we can conclude that $p_r^{se}-p_r^b < 0$ for all μ.

Part 2 (Online prices)

In comparing p_e^{se} and p_e^b, we have

$$p_e^{se}-p_e^b = \frac{1}{6}[3a-1+2\mu],$$

which is an increasing linear function of μ.

Case 1: $c \leq 1/2$. If $c \leq 1/2$, we have $\mu \in (L,U)$. Because $p_e^{se}-p_e^b$ is continuous in μ, we have

$$[p_e^{se}-p_e^b]\Big|_{\mu \to U} = [p_e^{se}-p_e^b]\Big|_{\mu = U} = \frac{a[1+c]}{3} > 0,$$
$$\text{and } [p_e^{se}-p_e^b]\Big|_{\mu \to L} = [p_e^{se}-p_e^b]\Big|_{\mu = L} = \frac{1}{6}[2ac+3a-1].$$

If $a \geq 1/[3+2c]$, we have that $p_e^{se}-p_e^b \geq 0$ at $\mu = L$. Because $p_e^{se}-p_e^b$ is increasing in μ, we have $p_e^{se}-p_e^b \geq 0$ for all $\mu \in (L,U)$. Otherwise if $a < 1/[3+2c]$, there is $\mu^* = [1-3a]/2$ such that if $\mu < \mu^*$, $p_e^{se} < p_e^b$; if $\mu > \mu^*$, $p_e^{se} > p_e^b$; if $\mu = \mu^*$, $p_e^{se} = p_e^b$.

Case 2: $c > 1/2$. If $c > 1/2$, we have $\mu \in (L, 1/2)$. At the upper bound,

$$[p_e^{se} - p_e^b]\Big|_{\mu \to \frac{1}{2}} = [p_e^{se} - p_e^b]\Big|_{\mu = \frac{1}{2}} = \frac{a}{2} > 0.$$

Similar to Case 1, we have that if $a \geq 1/[3+2c]$, $p_e^{se} - p_e^b \geq 0$ at $\mu = L$. Thus, $p_e^{se} - p_e^b \geq 0$ for all $\mu \in (L, 1/2)$. Otherwise if $a < 1/[3+2c]$, there is $\mu^* = [1-3a]/2$ such that if $\mu < \mu^*$, $p_e^{se} - p_e^b < 0$; if $\mu > \mu^*$, $p_e^{se} - p_e^b > 0$; otherwise if $\mu = \mu^*$, $p_e^{se} - p_e^b = 0$. Q.E.D.

Proof of Proposition 6 In comparing ω^{se} and ω^b, we have

$$\omega^{se} - \omega^b = \kappa_1 \mu^2 + \kappa_2 \mu + \kappa_3,$$

where

$$\kappa_1 = \frac{16a + 20}{72[a-1]}, \quad \kappa_1 = \frac{-72ac + 32a - 32}{72[a-1]}, \quad \text{and } \kappa_3 = \frac{36a^2c^2 - 72a^2c + 45a^2 + 72ac - 56a + 11}{72[a-1]}.$$

Thus, with respect to μ, $\omega^{se} - \omega^b$ is a concave parabola. Denoting the axis of symmetry as Ω, then we get

$$\Omega = -\frac{\kappa_2}{2\kappa_1} = \frac{9ac - 4a + 4}{4a + 5}.$$

When se and b models are considered together, from Definition 1, we get $\mu \in (L, \max\{1/2, U\})$. If $c \leq 1/2$, $U \leq 1/2$; if $c > 1/2$, $U > 1/2$.

Case 1: $c \leq 1/2$. If $c \leq 1/2$, we have $\mu \in \{L, U\}$. Because $\omega^{se} - \omega^b$ is continuous in μ, we have

$$\begin{aligned} [\omega^{se} - \omega^b]\Big|_{\mu \to U} &= [\omega^{se} - \omega^b]\Big|_{\mu = U} \\ &= \frac{1}{36} a \left[2a[1-2c]^2 - 12c + 15\right] \\ &\geq \frac{1}{36} a \left[-12 \times \frac{1}{2} + 15\right] > 0. \end{aligned}$$

$$[\omega^{se} - \omega^b]\Big|_{\mu \to L} = [\omega^{se} - \omega^b]\Big|_{\mu = L} = \frac{1}{72}\left[16a^2c^2 - 5a[8c-9] - 11\right].$$

When $a \in (0, 1)$ and $c \in (1/4, 1/2]$, we can derive that $\Omega > U$. Thus, $\omega^{se} - \omega^b$ is monotone increasing in μ when $\mu \in (L, U)$. Accordingly, we can conclude:

- If $16a^2c^2 - 5a[8c-9] - 11 \geq 0$, $\omega^{se} - \omega^b \geq 0$ at $\mu = L$, which means that $\omega^{se} - \omega^b > 0$ for all $\mu \in \{L, U\}$.
- If $16a^2c^2 - 5a[8c-9] - 11 < 0$, $\omega^{se} - \omega^b < 0$ at $\mu = L$, which means that there is a $\mu^* \in (L, U)$ such that

- if $\mu < \mu^*$, $\omega^{se} - \omega^b < 0$;
- if $\mu > \mu^*$, $\omega^{se} - \omega^b > 0$;
- if $\mu = \mu^*$, $\omega^{se} - \omega^b = 0$.

μ^* is the smaller solution of equation $\omega^{se} = \omega^b$; and,

$$\mu^* = \frac{-3\sqrt{-[a-1]\left[4a^2\left[4c^2-8c+5\right]+a[13-8c]+1\right]}+2a[9c-4]+8}{8a+10}.$$

Case 2: $c > 1/2$. When $c > 1/2$, $U > 1/2$ and, therefore, $\mu \in (L, 1/2)$. We can also derive that when $a \in (0,1)$ and $c > 1/2$, $\Omega > U$, which indicates that $\omega^{se} - \omega^b$ is monotone increasing in μ when $\mu \in (L, 1/2)$.

$$[\omega^{se} - \omega^b]\Big|_{\mu\to\frac{1}{2}} = [\omega^{se} - \omega^b]\Big|_{\mu=\frac{1}{2}} = \frac{a\left[4ac^2-8ac+5a+4c-4\right]}{8[a-1]}.$$

Thus, we can conclude:

- If $4ac^2 - 8ac + 5a + 4c - 4 \geq 0$, $[\omega^{se} - \omega^b]|_{\mu=1/2} \leq 0$, which indicates that $\omega^{se} - \omega^b < 0$ for all $\mu \in (L, 1/2)$.
- If $4ac^2 - 8ac + 5a + 4c - 4 < 0$, $[\omega^{se} - \omega^b]|_{\mu=1/2} > 0$, then we have
  - If $16a^2c^2 - 5a[8c-9] - 11 \geq 0$, $\omega^{se} - \omega^b \geq 0$ at $\mu = L$, which means that $\omega^{se} - \omega^b > 0$ for all $\mu \in \{L, 1/2\}$.
  - If $16a^2c^2 - 5a[8c-9] - 11 < 0$, $\omega^{se} - \omega^b < 0$ at $\mu = L$, which means that there is a $\mu^* \in (L, 1/2)$ such that
    - if $\mu < \mu^*$, $\omega^{se} - \omega^b < 0$;
    - if $\mu > \mu^*$, $\omega^{se} - \omega^b > 0$;
    - if $\mu = \mu^*$, $\omega^{se} - \omega^b = 0$.

    μ^* is the smaller solution of equation $\omega^{se} = \omega^b$; and,

$$\mu^* = \frac{-3\sqrt{-[a-1]\left[4a^2\left[4c^2-8c+5\right]+a[13-8c]+1\right]}+2a[9c-4]+8}{8a+10}.$$

Q.E.D.

Proof of Proposition 7 Using (16) and (24), we find $\gamma^{se} - \gamma^{br} = \frac{a}{9}(1 + c(10c-7))$. Therefore, if $(7-10c)c > 1$, then $\gamma^{se} < \gamma^{br}$. Combining constraints from (13) and (20) to obtain the range of $1/4 < c < 1/2$, the inequality is true for all $c \in (1/4, 1/2)$, and social costs are equal if $c = 1/2$. Q.E.D.

Proof of Proposition 8 In comparing γ^{se} and γ^b, we have

$$\gamma^{se} - \gamma^b = \kappa_1 \mu^2 + \kappa_2 \mu + \kappa_3,$$

where

$$\kappa_1 = \frac{80a + 28}{72[a-1]}, \quad \kappa_2 = \frac{[-216ac + 16a - 16]}{72[a-1]},$$
$$\text{and } \kappa_3 = \frac{108a^2c^2 - 72a^2c + 9a^2 + 72ac - 10a + 1}{72[a-1]}.$$

Thus, with respect to μ, $\gamma^{se} - \gamma^b$ is a concave parabola. The axis of symmetry, denoted by Ω, is

$$\Omega = \frac{27ac - 2a + 2}{20a + 7}.$$

When se and b models are considered together, we have $\mu \in (L, \min\{1/2, U\})$. Accordingly, if $c \leq 1/2$, $U \leq 1/2$; otherwise, if $c > 1/2$, $U > 1/2$.

If $c \leq 1/2$, $U \leq 1/2$ which means $\mu \in (L, U)$. Because $\pi_r^{se} - \pi_r^b$ is continuous in μ, we have

$$\begin{aligned} [\gamma^{se} - \gamma^b]\Big|_{\mu \to U} &= [\gamma^{se} - \gamma^b]\Big|_{\mu = U} \\ &= \frac{1}{18} a[2c-1][10ac - 5a + 3] \\ &< \frac{1}{18} a[2c-1][10a \times \frac{1}{4} - 5a + 3] \\ &< 0 \end{aligned}$$

$$\text{and } [\gamma^{se} - \gamma^b]\Big|_{\mu \to L} = [\gamma^{se} - \gamma^b]\Big|_{\mu = L} = \frac{1}{72}\left[80a^2c^2 - 56ac + 9a - 1\right].$$

When $a \in (0,1)$ and $c \in (1/4, 1/2]$, we have $80a^2c^2 - 56ac + 9a - 1 < 0$, which means $\gamma^{se} - \gamma^b < 0$ when $\mu = L$. We can also show $L < \Omega < U$ when $a \in (0,1)$ and $c \in (1/4, 1/2]$. At $\mu = \Omega$, we have

$$[\gamma^{se} - \gamma^b]\Big|_{\mu = \Omega} = \frac{20a^2\left[12c^2 - 8c + 1\right] + a[3 - 8c] + 1}{8[20a + 7]}$$

Hence, we can conclude:

- If $[\gamma^{se} - \gamma^b]|_{\mu = \Omega} \leq 0$, $\gamma^{se} - \gamma^b \leq 0$ for all $\mu \in (L, U)$.
- If $[\gamma^{se} - \gamma^b]|_{\mu = \Omega} > 0$, there are μ_1^* and μ_2^* in (L, U), $\mu_1^* < \mu_2^*$, such that
  - if $\mu_1^* < \mu < \mu_2^*$, $\gamma^{se} - \gamma^b > 0$;
  - if $\mu < \mu_1^*$ or $\mu > \mu_2^*$, $\gamma^{se} - \gamma^b < 0$;
  - if $\mu = \mu_1^*$ or $\mu = \mu_2^*$, $\gamma^{se} - \gamma^b = 0$.

When $c > 1/2$, $U > 1/2$ which means $\mu \in (L, 1/2)$. It is difficult to definitively characterize the difference in π_r^b and π_r^{se}. However, the general trend continues. Q.E.D.

Proof of Proposition 9 In comparing γ^{br} and γ^{b}, we have

$$\gamma^{br} - \gamma^{b} = \kappa_1 \mu^2 + \kappa_2 \mu + \kappa_3,$$

where

$$\kappa_1 = \frac{80a+28}{72[a-1]}, \quad \kappa_2 = \frac{16a-16}{72[a-1]}, \quad \text{and } \kappa_3 = \frac{28a^2c^2 - 16a^2c + a^2 + 80ac^2 - 200ac - 2a + 1}{72[a-1]}.$$

Thus, with respect to μ, $\gamma^{br} - \gamma^{b}$ is a concave parabola. Since $\gamma^{br} - \gamma^{b}$ is continuous in μ, we get

$$[\gamma^{br} - \gamma^{b}]\Big|_{\mu \to U'} = [\gamma^{br} - \gamma^{b}]\Big|_{\mu = U'} = \kappa_1' c^2 + \kappa_2' c + \kappa_3',$$

where

$$\kappa_1' = \frac{[20a^3 + 150a^2 + 288a + 28]}{162[a-1]}, \quad \kappa_2' = \frac{[-20a^3 - 51a^2 - 405a - 10]}{162[a-1]},$$

$$\text{and } \kappa_3' = \frac{5a^3 - 12a^2 + 9a - 2}{162[a-1]}.$$

Observe that $[\gamma^{br} - \gamma^{b}]|_{\mu \to U'}$ is a concave function of c. Thus, we only need $[\gamma^{br} - \gamma^{b}]|_{\mu \to U'} > 0$ when $c = 1/8$ and $c = 1/2$ to show $[\gamma^{br} - \gamma^{b}]|_{\mu \to U'} > 0$ for all $c \in (1/8, 1/2)$. Because

$$[\gamma^{br} - \gamma^{b}]\Big|_{\mu = U', c = \frac{1}{8}} = \frac{10[a^3 - 1] - 57a^2 - 132a}{576[a-1]} > 0$$

$$\text{and } [\gamma^{br} - \gamma^{b}]\Big|_{\mu = U', c = \frac{1}{2}} = \frac{-3a}{4[a-1]} > 0,$$

we can conclude that $\gamma^{br} > \gamma^{b}$ when $\mu \to U'$ for all $a \in (0,1)$ and $c \in (1/8, 1/2)$.

At the lower bound of μ, we have

$$[\gamma^{br} - \gamma^{b}]\Big|_{\mu = L} = \frac{80a^3c^2 + 56a^2c^2 + a^2 + 80ac^2 - 216ac - 2a + 1}{72[a-1]}.$$

Since $\gamma^{br} - \gamma^{b}$ is concave in μ and $[\gamma^{br} - \gamma^{b}]|_{\mu = U'} > 0$, we can conclude:

- If $[\gamma^{br} - \gamma^{b}]|_{\mu = L} \geq 0$, then $\gamma^{br} - \gamma^{b} > 0$ for all $\mu \in (L, U')$.
- If $[\gamma^{br} - \gamma^{b}]|_{\mu = L} < 0$, then there is μ^* in (L, U') such that
  - if $\mu < \mu^*$, $\gamma^{br} - \gamma^{b} < 0$;
  - if $\mu > \mu^*$, $\gamma^{br} - \gamma^{b} > 0$;
  - if $\mu = \mu^*$, $\gamma^{br} - \gamma^{b} = 0$.

  μ^* is the smaller solution of $\gamma^{br} = \gamma^{b}$; and,

$$\mu^* = \frac{-\sqrt{[16a-16]^2 - 4[80a+28]\,[28a^2c^2 - 16a^2c + a^2 + 80ac^2 - 200ac - 2a + 1]} - 16a + 16}{8[20a+7]}.$$

Q.E.D.

How An Online Entry Model Can Lead to Market Structures We Do Not See In Practice

We make the point that market structures we see in practice are a result of twenty years of e-commerce where there has been dramatic changes in technology such as search, visualization, broadband penetration and now mobile e-commerce. We take the positive economics approach and model market structures we see in practice: a dominant e-tailer as per Balasubramanian's model (b), the Salop model with online stores (se) and the Balasubramanian model with retailers selling online (br). What we rarely see in practice is substantial retail stores without an online presence – for example, even a retailer like B&H Photo in New York City with one SuperStore has a significant online presence to compete with national photo shop chains. We also rarely see a single traditional retailer with an online presence as evidenced by our Figure 1 in the article where we see multiple traditional retailers online in every NAICS 3-digit subsector of the retail economy.

To show how an online entry model can lead to market structures we do not see in practice and may not lead to market structures we commonly see in practice, we set up a three-stage sequential entry game. In this parsimonious and standard entry game the only history we include is implicit in that the sequence of entry is predetermined. A more elaborate characterization of history would essentially pre-ordain the result, and our goal is presumably not to find a model that produces our desired result. The sequence in our online entry model is as follows:

- Stage 1: Retailer A decides whether to open an online store;
- Stage 2: Retailer B decides whether to open an online store;
- Stage 3: The pure e-tailer decides whether to enter online.

After Stage 3 the retailers and e-tailer set prices simultaneously as in our models. There are six possible outcomes, although not all of them are necessarily equilibria:

- the Balasubramanian model with retailers selling online (br),
- the Salop model with online stores (se),
- the Balasubramanian model (b)
- the Salop model (s),
- the Balasubramanian model with one retailer selling online (ba)
- the Salop model with one retailer selling online (sa)

Profits for the first four market structures are detailed in the article. It remains to determine profits under the last two.

Salop model with one retailer selling online (*sa*) We take retailer A to be the dual-channel retailer and retailer B to be the traditional store. In the online store, retailer A offers the identical good at $p_{Are} + \mu - ac + ax$. We define two indifferent consumers. The first, x_1, is indifferent between the retailer A's traditional and online stores. This indifferent consumer is defined by $p_{Ar} + tx_1 = p_{Are} + \mu - ac + ax_1$, giving the distance away from retailer A's traditional retail store as $x_1 = [ac + p_{Ar} - p_{Are} - \mu]/[a - t]$. The second, x_2, is indifferent between the retailer A's online store and retailer B's traditional store. This indifferent consumer is defined by $p_{Are} + \mu - ac + ax_2 = p_B + t(1/2 - x_2)]$.

Consequently, retailer A's profit maximization problem is

$$\begin{aligned}\max_{p_{Ar},p_{Are}} \pi_A^{sa} &= \max_{p_{Ar},p_{Are}} \left\{ p_{Ar} \left[2\frac{ac + p_{Ar} - p_{Are} - \mu}{a - t} \right] \right. \\ &\left. + 2p_{Are} \left[\frac{2ac - 2p_{Are} + 2p_{Br} - 2\mu + t}{2(a+t)} - \frac{ac + p_{Ar} - p_{Are} - \mu}{a - t} \right] \right\}.\end{aligned}$$

Retailer B's profit maximization problem is

$$\max_{p_{Br}} \pi_B^{sa} = \max_{p_{Br}} \left\{ 2p_{Br} \left[\frac{1}{2} - \frac{2ac - 2p_{Are} + 2p_{Br} - 2\mu + t}{2(a+t)} \right] \right\}.$$

The resulting Nash equilibrium prices are

$$p_{Ar}^{sa} = [a - ac + \mu + 2t]/6, \quad p_{Are}^{sa} = [a + 2ac + 2t - 2\mu]/6, \text{ and } p_{Br}^{sa} = [2a(1-c) + 2\mu + t]/6,$$

all of which are positive from (2) in the article and when both channels of retailer A have positive market share, i.e., $a^2c + 5act + 2t^2 > a^2 + a(\mu + t) + 5\mu t$.

Retailer profits are

$$\begin{aligned}\pi_A^{sa} &= \frac{a^3((4-5c)c+1) + a^2(2(5c-2)\mu + ((4-13c)c+3)t)}{18(a-t)(a+t)} \\ &\quad \frac{-a(2ct(4t-13\mu) + \mu(5\mu+4t)) + t\left(-13\mu^2 - 4t^2 + 8\mu t\right)}{18(a-t)(a+t)}\end{aligned}$$

and

$$\pi_B^{sa} = \frac{(-2a(c-1) + 2\mu + t)^2}{18(a+t)}.$$

Balasubramanian model with one retailer selling online (*ba*) As in the *sa* model, we take retailer A to be the dual-channel retailer and retailer B to be the traditional store. In addition, we have a pure e-tailer selling online. In the online store, retailer A offers the identical good at $p_{Are} + \mu - ac + ax$. The pure e-tailer charges $p_e + \mu$ for the same good. We define three indifferent consumers. The first, x_1, is indifferent between retailer A's traditional and online stores, and is defined by $p_{Ar} + tx_1 = p_{Are} + \mu - ac + ax_1$, giving the distance away from the traditional store as $x_1 = [ac + p_{Ar} - p_{Are} - \mu]/[a - t]$. The second, x_2, is indifferent between the retailer A's online store and the pure etailer, and is defined by $p_{Are} + \mu - ac + ax_2 = p_e + \mu$. Finally, the third consumer, $x3$ is indifferent between the pure e-tailer and retailer B's traditional store, and is defined as $p_{Br} + \mu - ac + ax_2 = p_e + \mu$.

Consequently, retailer A's profit maximization problem is

$$\begin{aligned}\max_{p_{Ar},p_{Are}} \pi_A^{ba} &= \max_{p_{Ar},p_{Are}} \left\{ 2p_{Ar}\left[\frac{ac + p_{Ar} - p_{Are} - \mu}{a-t}\right]\right. \\ &\left. +2p_{Are}\left[\frac{ac - p_{Are} + p_e}{a} - \frac{ac + p_{Ar} - p_{Are} - \mu}{a-t}\right]\right\}.\end{aligned}$$

Retailer B's profit maximization problem is

$$\max_{p_{Br}} \pi_B^{br} = \max_{p_{Br}} \left\{2p_{Br}\left[\frac{p_e - p_{Br} + \mu}{t}\right]\right\}.$$

The e-tailer's profit maximization problems is

$$\max_{p_e} \pi_e^{br} = \max_{p_e} \left\{2p_e\left[\frac{2p_{Br} - 2p_e - 2\mu + t}{2t} - \frac{ac - p_{Are} + p_e}{a}\right]\right\}.$$

The resulting Nash equilibrium prices are

$$\begin{aligned} p_{Ar}^{ba} &= \frac{a(t - ct + 2\mu) + 3\mu t}{6(a+t)}, & p_{Are}^{ba} &= \frac{a(3ac + 2ct - \mu + t)}{6(a+t)} \\ p_{Br}^{ba} &= \frac{a(t - ct + 2\mu) + 3\mu t}{6(a+t)}, \quad \text{and} & p_e^{ba} &= \frac{a((1-c)t - \mu)}{3(a+t)},\end{aligned}$$

all of which are positive from (2) in the article and when $t(1-c) \geq \mu$.

Retailer profits are

$$\begin{aligned}\pi_A^{ba} \quad = \quad & \frac{2a\mu(6a^2c + a(19c-1)t + (11c+1)t^2) + at(3a^2(2-5c)c}{18(a-t)(a+t)^2} \\ & \frac{+a(1 - c(17c+2))t - (2ct+t)^2) + \mu^2(-(8a^2 + 19at + 9t^2))}{18(a-t)(a+t)^2},\end{aligned}$$

$$\pi_B^{ba} = \frac{(a(c-1)t - 2a\mu - 3\mu t)^2}{18t(a+t)^2}, \quad \text{and} \quad \pi_e^{ba} = \frac{2a((c-1)t + \mu)^2}{9t(a+t)}.$$

Profit relations The relations between retail profits from our article and the additional market structures above for retailers A and B are

$$\pi^s > \pi_A^{sa} > \pi_B^{sa} > \pi^{se}, \quad \pi_r^b, \pi_A^{ba} \geq \pi_B^{ba}, \pi_r^{br}, \quad \pi_A^{sa} > \pi_r^b \;\text{ and }\; \pi^{se} > \pi_B^{ba}$$

The relations between e-tailer profits from our article and additional market structures are

$$\pi_e^b > \pi_e^{ba} > \pi_e^{br}. \tag{A1}$$

Entry Game We work backwards through the stages to determine the feasible market structures. We also incorporate a fixed cost of pure-e-tailer entry, consistent with a pure e-tailer needing to set up inventory and distribution. This creates four possible scenarios for pure e-tailer entry given the profit relations between the six possible market structure outcomes. The four possible scenarios depend on which elements in (A1) are positive.

Net of fixed cost of pure-e-tailer entry, $\pi_e^b < 0$*.* The extensive form game tree is shown in Figure 1 below. When fixed cost of pure-e-tailer entry are high, the pure-e-tailer does not enter, and the equilibrium of the entry game is the Salop model (s).

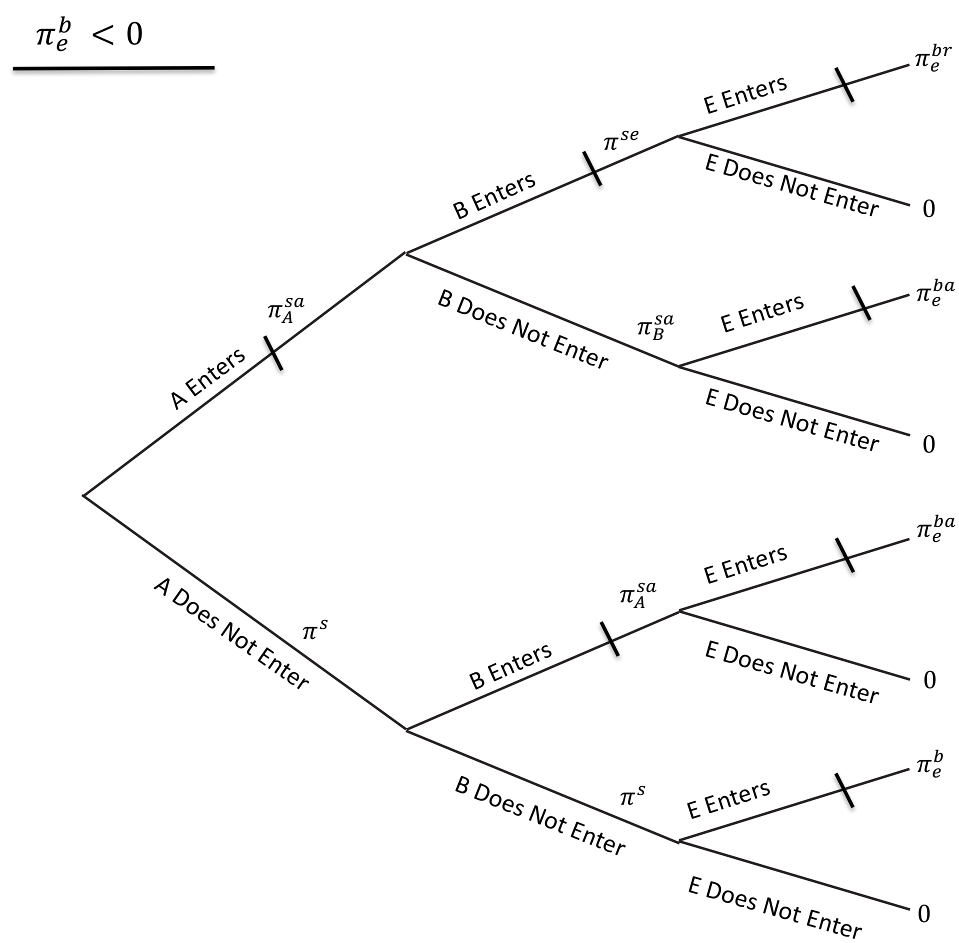

Figure 1: Entry game: high fixed cost of pure-e-tailer entry

Net of fixed cost of pure-e-tailer entry, $\pi_e^b > 0 > \pi_e^{ba}$*.* The extensive form game tree is shown in Figure 2 below. When fixed cost of pure-e-tailer entry are medium, the pure-e-tailer only enters in one branch (at the bottom of Figure 2), and the equilibrium of the entry game is the Salop model with one retailer selling online (sa).

Net of fixed cost of pure-e-tailer entry, $\pi_e^{ba} > 0 > \pi_e^{br}$*.* The extensive form game tree is shown

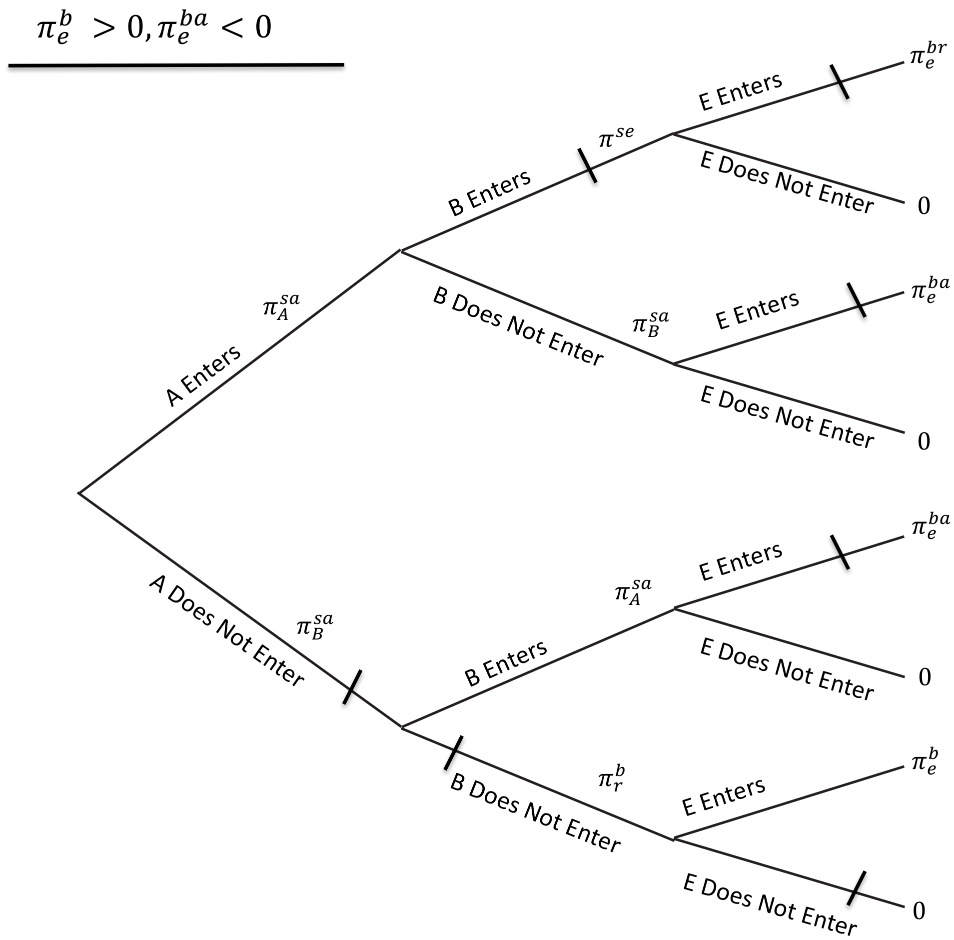

Figure 2: Entry game: medium fixed cost of pure-e-tailer entry

in Figure 3 below. When fixed cost of pure-e-tailer entry are low, the pure-e-tailer only enters in two branches (the lower half of Figure 3). The equilibrium of the entry game can be one of three market structures. If the pure e-tailer does not enter, then the equilibrium is the Salop model with online stores (*se*). If the pure e-tailer does enter, then the equilibrium is either the Balasubramanian model (*b*) or the Balasubramanian model with one retailer selling online (*ba*).

Net of fixed cost of pure-e-tailer entry, $\pi_e^{br} > 0$*.* The extensive form game tree is shown in Figure 4 below. When fixed cost of pure-e-tailer entry are zero, the pure-e-tailer enters in each branch in Stage 3 (the right-hand side of Figure 4). The equilibrium of the entry game can be one of two market structures, a subset of that in Figure 3: the equilibrium is either the Balasubramanian model (*b*) or the Balasubramanian model with one retailer selling online (*ba*).

Summary The most likely setting in practice has low or negligible fixed cost of pure-e-tailer entry relative to pure e-tailer profits. This corresponds to the entry game scenarios shown

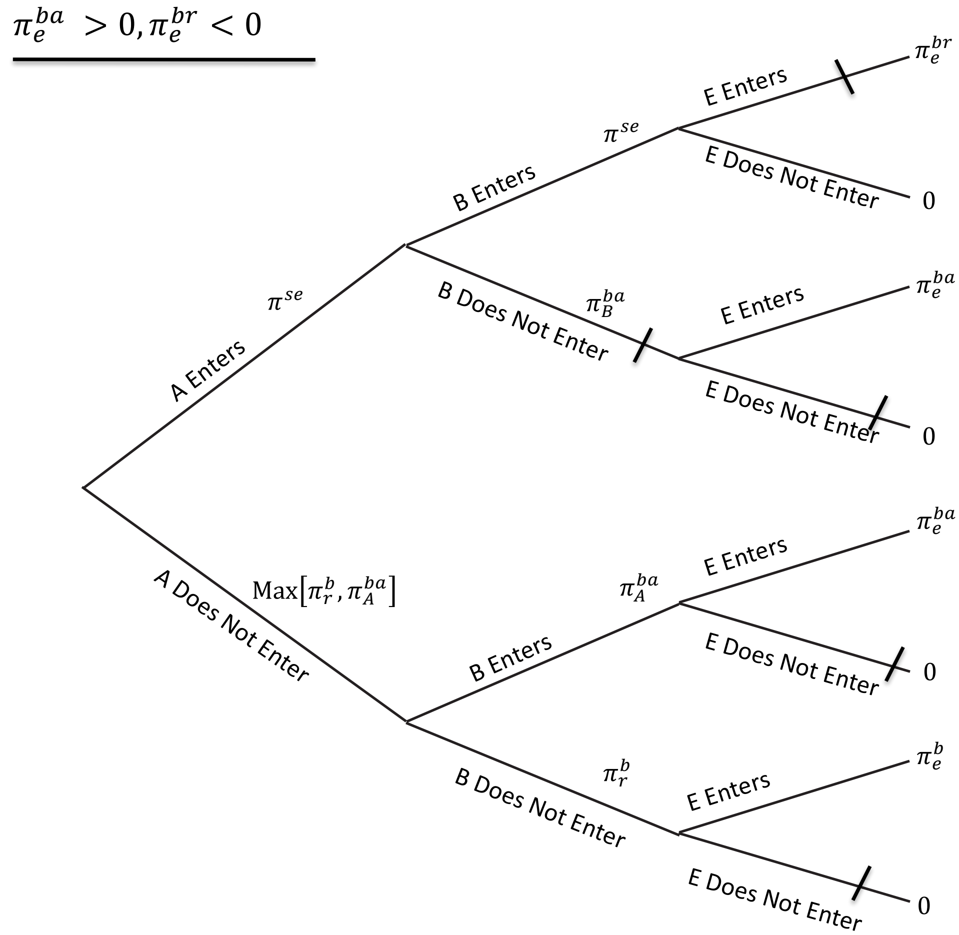

Figure 3: Entry game: low fixed cost of pure-e-tailer entry

in Figures 3 and 4. The possible equilibria in these scenarios depends on the settings of the parameters in our underlying model: unit transportation cost t, fixed online disutility cost μ, the marginal drop in mitigation of online disutility costs a, and the maximum amount of mitigation of online disutility costs, ac. Of the possible equilibria in these game scenarios, two of three are market structures we study.

It is worth noting that equilibria of the entry game scenarios include two market structures we rarely see in practice, those with a single traditional retailer selling online. In contrast, a common market structure we see in practice, multiple traditional retailers selling online together with a pure e-tailer like Amazon.com (our br market structure) is not an equilibrium in the entry game. The entry game scenario where br could be an equilibrium is in Figure 4, where the intensity of online competition causes one or both traditional retailers to have been more profitable staying offline, and in the sequential game they do not enter. What we observe in practice might result from an entry game model if a more elaborate characterization of history such as asymmetries

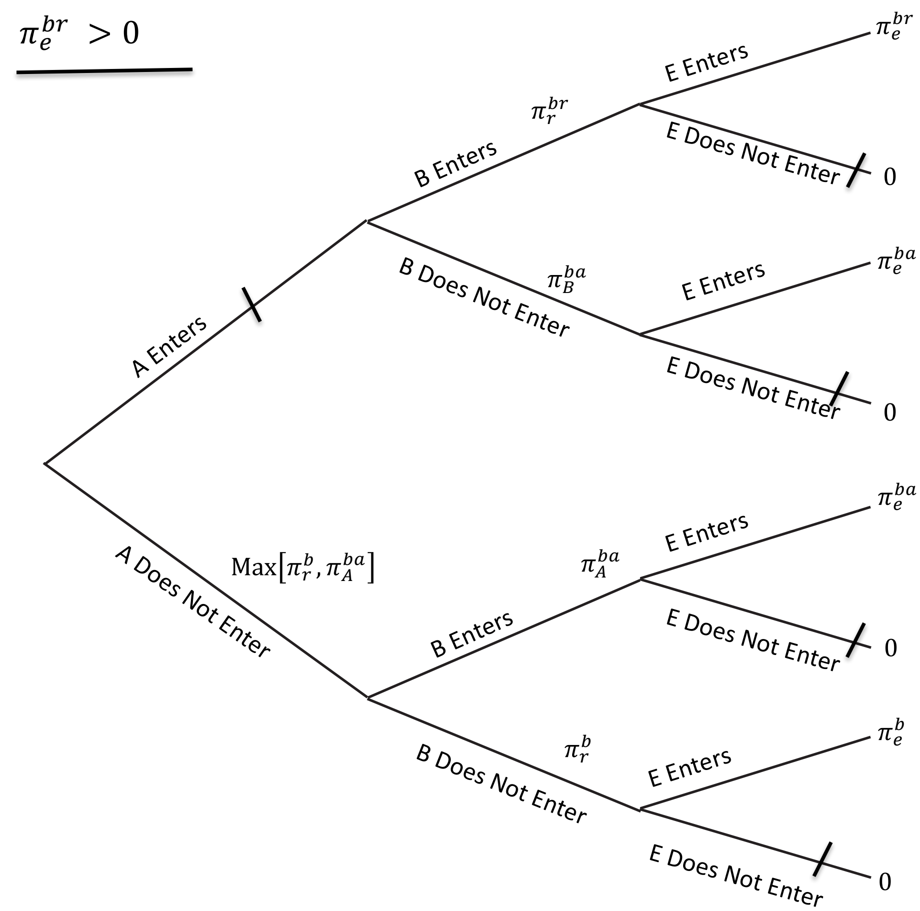

Figure 4: Entry game: zero fixed cost of pure-e-tailer entry

in online fixed costs or marginal costs of online order processing and distribution for traditional retailers, which may change over time, or additional dimensions of differentiation, were included in the formulation. Elements such as these would not only cause the retailers to be asymmetric both offline and online, but also cause their impacts on the intensity of price competition with an e-tailer to differ.

Absent a formulation that characterizes history in such a way as to simply produce the desired equilibrium to justify examining the equilibrium, there is a role for the positive economics approach of developing knowledge about "what is" when the complexity of history is too elaborate to be parsimoniously included in a model that leads to market structures that in practice are "what is" without being overly contrived.